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The dynamic response of SDOF sysytems loaded by a shock wave

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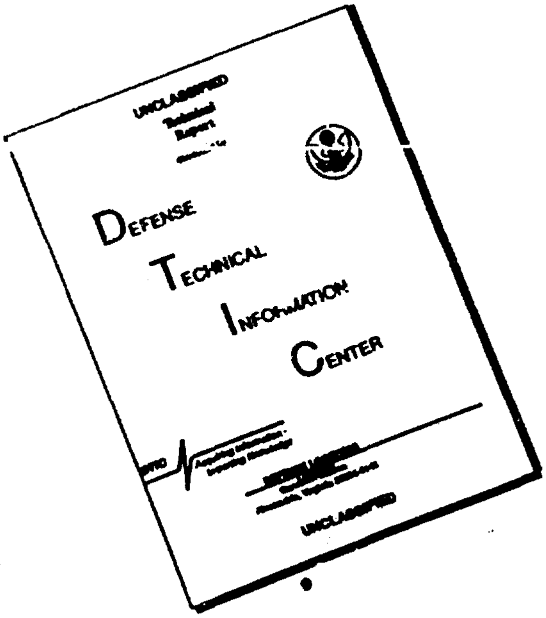

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The dynamic response of SDOF sysytems loaded by a shock wave

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Summary

To perform a quick analysis or a parameter study of structures under dynamic loads, the structure is often regarded as a Single Degree of Freedom (SDOF) system with one characteristic deflection. At the TNO Prins Maurits Laboratory these systems are used as tools for the vulnerability analysis of the structures of weapon platforms under explosion loads. For naval ships, SDOF techniques are applied for the internal blast code DAMINEX, for the external blast algorithm CBD, for ship door research and for the research of underwater shock on the hull. Therefore research is done on SDOF systems, particularly for the response of stiffened panels under large deflections introducing non linear mechanisms.
This report gathers and explains some general techniques of structural dynamics for SDOF systems and serves as a basis for the application of the SDOF technique in vulnerability research. Analytical formulae for the maximum dynamic deflection have been determined, where the deflections are expressed in general terms for force, mass and stiffness. A general expression for the Dynamic Load Factor (DLF) is established for linear and non-linear deformation characteristics. Convenient mathematical approximations have been found for the response curves. New transformations factors have been derived. The curve for iso-damage based on the Pressure-Impulse technique is considered as a tool for the scope beyond the DLF.
Parameter studies can be easily performed with the obtained results.

Samenvatting

Het één-graad-van-vrijheid model (SDOF) met één karakteristieke verplaatsing, wordt vaak toegepast wanneer een snelle analyse van een constructie onder dynamische belasting gewenst is.
Op het Prins Maurits Laboratorium worden deze systemen gebruikt als gereedschappen voor het kwetsbaarheidsonderzoek van constructies van wapenplatformen onder explosiebelasting. Bij marineschepen worden SDOF-technieken toegepast voor de inwendige blastcode DAMINEX, het uitwendige blast-algoritme CBD, voor het scheepsdeurmodel en voor het onderzoek naar onderwaterschok op de huid. Daartoe wordt er onderzoek verricht naar SDOF-systemen, in het bijzonder ten behoeve van de respons van verstijfde panelen bij grote verplaatsingen, die niet lineair gedrag introduceren.
Dit rapport inventariseert en verklaart enkele algemene technieken van de constructiedynamica voor SDOF-systemen en dient als een basis voor de toepassing van de SDOF-techniek in het kwetsbaarheidsonderzoek. Analytische formules zijn bepaald voor de maximale dynamische verplaatsing,

waarbij de verplaatsingen zijn uitgedrukt in algemene termen zoals kracht, massa en veerstijfheid. Een algemene uitdrukking voor de dynamische belastings (DLF) factor is bepaald voor zowel lineaire als niet-lineaire vervormingskarakteristieken. Eenvoudige mathematische benaderingen zijn bepaald voor responsgrafieken. Nieuwe transformatiefactoren zijn afgeleid. De iso-schadegrafiek zoals gebaseerd op de Druk-Impuls techniek is onderzocht als middel buiten het geldigheidsgebied van de DLF.

Met behulp van de verkregen resultaten kunnen de invloeden van de verschillende parameters eenvoudig onderzocht worden.

CONTENTS

LIST OF SYMBOLS

F	=	force	[N]
x	=	deflection	[m]
k	=	stiffness	$[\frac{N}{m^n}]$
n	=	exponent of deformation characteristic	[-]
W	=	work done	[J]
E	=	energy	[J]
U	=	internal energy	[J]
m	=	mass	[kg]
t	=	time	[s]
p	=	pressure	[Pa]
θ	=	time constant	[s]
A	=	surface	[m^2]
I	=	impulse	[N.s]
M	=	momentum	[kg.m/s]
C	=	constant	
T	=	natural vibration period	[s]
ω	=	natural circular frequency	[rad/s]
DLF	=	Dynamic Load Factor	[-]
f	=	natural frequency	[Hz]
v	=	velocity	[m/s]
a,b,c	=	fit constants	
l	=	length	[m]
w	=	width	[m]
t*	=	equivalent thickness	[m^3/m^2]
Tf,Tk,Tm	=	transformation factors	[-]
u	=	distance coordinate in length direction	[m]
EI	=	bending stiffness of a beam	[$N.m^2$]
α	=	time factor	[-]
tp	=	phase duration of shock wave	[s]
s	=	shape factor of shock wave	[-]
fx(u)	=	deflection shape as function of distance coordinate	[-]
fp(u)	=	pressure distribution as function of distance coordinate	[-]

fx(t)	=	centre deflection history as function of time	[-]
DN	=	damage number	
RF	=	Reflection factor	[-]

<u>subscripts</u>

m	-	maximum
ms	-	static
c	-	according to centre
be	-	bending elastic
bemax	-	bending elastic maximum
bp	-	bending plastic
me	-	membrane elastic
memax	-	membrane elastic maximum
mp	-	membrane plastic
mpmax	-	membrane plastic maximum
e	-	external
i	-	internal
d	-	inertia
r	-	retarding
p	-	potential
k	-	kinetic
s	-	specific value (divided by surface area)
f	-	free field or incident property of shock wave
rel	-	relative
+	-	just more
beam	-	concerning beams (1D structures)
crit	-	critical value

1 INTRODUCTION

The response of structures loaded by a dynamic (transient and/or steady state) load is often considered as a Single Degree Of Freedom (SDOF) problem or a one-spring-mass system. This simple approach enables the scientist to perform a quick analysis and to gain insight into the influence of the parameters.
In the Weapon Effectiveness group of the TNO Prins Maurits Laboratory, SDOF models are used to analyze the vulnerability of structures to blast loadings originating from internally or externally detonating warheads in air or water. The structure can be a part of weapon platforms like aircraft, shelters and ships. For ships, the SDOF technique is applied for the internal blast code DAMINEX, for the external blast algorithm CBD, for ship door research and for the research of underwater shock on the hull.

Although many aspects of SDOF systems are described in (Biggs, 1964) and (Baker, 1979), research continues on these SDOF systems particularly for the application on stiffened panels. The reason is that such structures subjected to explosion loads are commonly related to large deflections which introduce the non-linear membrane mechanism besides the regular bending mechanism.

The purpose of this report is to investigate and rank some general aspects of structural dynamics for SDOF systems. Furthermore the SDOF technique is applied for some commonly used structural elements.
The structural response is expressed in general terms for force, stiffness and mass for stiffened panels to study parameter influence. Analytical formulae will be derived for the maximum dynamic response. The user is made aware of the situations in which the derived formulae can be applied. Furthermore the relation between the maximum dynamic deflection and the static one is examined via the so-called Dynamic Load Factor (DLF). This factor is very important, as the designer is mostly familiar only with static loads. This factor has to be extended to non-linear mechanisms. Finally, the relation of SDOF systems with the so-called Pressure Impulse (P-I) technique is investigated. The P-I technique is used for the external blast CBD algorithm and is a suitable tool for vulnerability assessments of structures and components.
In a report to be published soon (Keizers, Erkel, 1992), the research on SDOF solutions for the bending and membrane mechanism of stiffened panels will be given more specifically.
The response model based on SDOF equations for the stiffened panels will be validated by the Dutch 'Roofdier' trials and by a comparison with FEM results.

2 DEFORMATION CHARACTERISTICS FOR STATIC LOADS

In structural analyses, a structure is often generalized to commonly known elements, such as beams, plates and shells etc. If a static load F_{ms} is applied, a plate, for example, will deflect or deform into a certain shape. The deflection in an arbitrary position of the plate can be related to the centre deflection x_m by means of a deflection shape, see Figure 1. This deflection shape is influenced by the load distribution and the boundary conditions. For dynamic loads, the shape also depends on the pressure-time history and the mass distribution. The complete solution for these systems involves multi degrees of freedom. By employing an approximate deflection shape, the response problem for structures can be made a single degree of freedom SDOF. If the structure is composed of several parts each one having its own chosen deflection shape, the solution again becomes a multi degree of freedom.

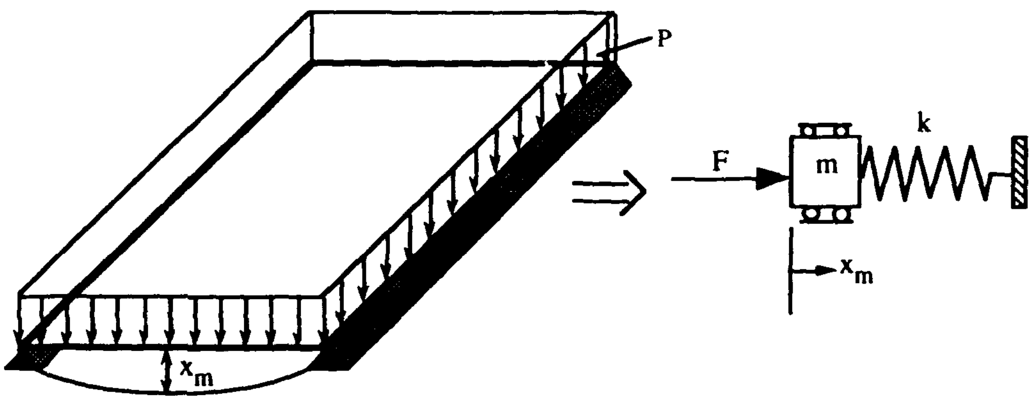

Figure 1 Transformation of structure into a one spring mass system

For structures under dynamic load, the first modal shape can be used as an approximate deflection shape to deal with the actual structure as a SDOF structure. But on many occasions a more simple deflection shape is chosen because modal shapes cannot always easily be determined or can only be expressed by unmanageable mathematical functions. So, often it is assumed that the structure will take the same deflection shape as in a static condition under a uniformly distributed load or as a simple harmonic function.

For the transformation of the actual structure, which is first assumed as a SDOF system by means of one deflection shape, into a one-spring-mass system with a SDOF, see Figure 1, transformation factors have to be applied, see paragraph 3.5.

In this report, uniformly distributed loads will be emphasized, i.e. structural elements loaded by a pressure which does not vary over the surface of the element. However, the presented solutions will not change due to non-uniform pressure distributions, only the distribution of the pressure must be input into the transformation factors.

For structural elements, a typical load-deflection (F_{ms}-x_m) or deformation characteristic can be defined. This is the expression which relates the static load (F_{ms}) to the static centre deflection (x_m) via the stiffness of the structure:

$$F_{ms} = x_m{}^n \cdot k \tag{1}$$

where n is the exponent of the deformation characteristic and k is the stiffness or spring constant of the structure.

Now some deformation characteristics will be considered, where the exponent n has a certain discrete value. A plate or a beam can develop two resistance mechanisms, the well-known bending mechanism and the membrane mechanism. Both mechanisms will start elastically and will become plastic for larger deformations (x_m). Notice that sometimes these deformation characteristics can occur together.

Bending

First the bending mechanism is considered which is based on shear forces and bending moments in the lateral direction of the loaded structure. Two parts can be distinguished in this mechanism, see Figure 2:

1) linear elastic deformation (elastic bending beam), n =1.
2) retarding force or plastic deformation (e.g. complete plastic bending beam), n = 0.

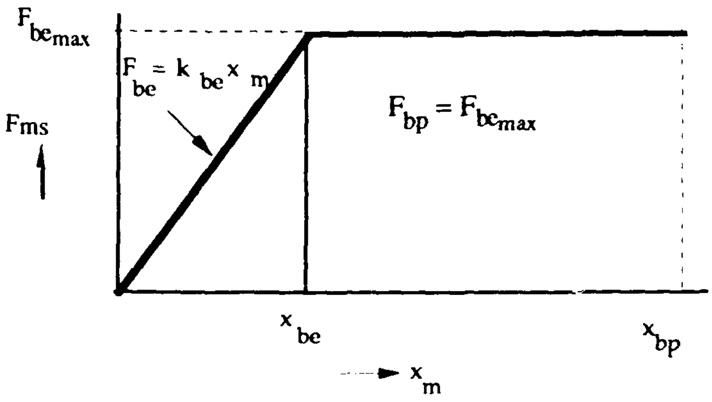

Figure 2 Elastic and plastic bending deformation characteristics

In the case of a plastic bending beam there is a non-linearity due to material properties; a so-called physical non-linearity.

Membrane

Secondly, the membrane mechanism is considered which also consists of two parts. This mechanism is based on longitudinal stretching of the panel, which is induced due to axial non-movable or partly movable supports. It is a geometrically non-linear mechanism. The deformation characteristics can be derived by determining the deformation energy for a thin plate and are shown in Figure 3. For a formal derivation and formulae see (Keizers, Erkel, 1992).

3) elastic membrane, n = 3.

4) plastic membrane, n = 1.

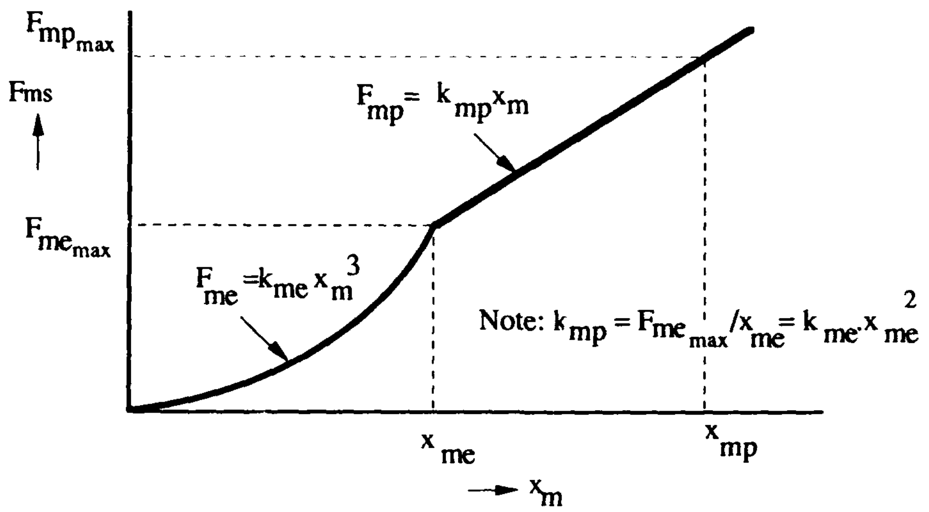

Figure 3 Elastic and plastic membrane deformation characteristics

The stiffnesses k are different in magnitude and dimension. They depend upon length, stiffness, cross-section, yield limit etc, and can consist of several terms.

It should be noted that in this report all formulae are derived with respect to the maximum deflection, but in fact the maximum strain is more interesting for determining the remaining capacity of a structure. Although the strain can be always related to the deflection, it is not proportional to deflection. From deflection - strain relations it is shown that for the bending deformation characteristic the strain is nearly linearly proportional to x_m, while for the membrane deformation characteristic the strain is proportional to x_m^2.

3 DYNAMIC STRUCTURAL RESPONSE

The main difference between static and dynamic response is that during dynamic loading the applied load varies with time in such a way that the arising inertia forces on the structure cannot be neglected. As a consequence, the centre deflection, x_c, is time dependent with a maximum value of x_m:

$$x_c = x_m \cdot fx(t)$$

3.1 General principles

In the following paragraphs some general techniques will be discussed as a basis for SDOF response.

3.1.1 The energy equation

According to the first law of thermodynamics, if energy is supplied to a system this energy contributes to the internal energy of that system. In this report the system is bound by a conservative spring, a non-conservative spring and the mass. The energy supply related to the boundary of the system consists of work done by external forces (W_e) only.

So, $W_e = \Delta$(internal energy). Furthermore the internal energy is defined in terms of kinetic energy of the mass, and deformation energy of the springs, while the starting internal energy is set to zero. Evaluating this, the conservation equation of energy, which is valid at any time of the response, can be written as:

$$W_e = U + E_k \tag{2}$$

where: W_e : work done is the line integral of the applied load over the covered way = $\int F_e(t)dx_c$

E_k : kinetic energy[1] of structure = $\frac{1}{2} m \left[\frac{dx_c}{dt}\right]^2$

U : deformation energy composed of potential energy and/or heat increase.

---

[1] Note that kinetic energy is a positive scalar while work done depends on the history and can be negative.

The kinetic energy can be calculated by the opposite of the work done by the inertia forces[1]: $E_k = -\int F_d dx_c$

The potential energy can be calculated by the work done by the conservative forces (elastic) on the spring: $\int F_p dx_c$

The heat increase can be calculated by the work done by non-conservative forces (plasticity, friction etc.) on the spring : $\int F_r dx_c$.

Both terms of the deformation energy are combined here in one term which can be calculated by the work done by the internal taken force of an arbitrary spring type which can consist of several deformation or velocity characteristics: $U = \int F_i dx_c = \int F_p dx_c + \int F_r dx_c$.

In this report, only undamped systems are regarded because for real structures the damping will have only minor effects on the first maximum response. Therefore velocity characteristics are not considered.

Parameters are defined as:

- F_e : external forces (surface force)
- F_d : inertia forces (body force)
- F_p : potential forces taken by the structure
- F_r : retarding forces taken by the structure
- F_i : internal force taken by the structure
- m : lumped mass
- x_c : centre deflection

3.1.2 Deformation energy

The deformation energy is defined as:

$$U = \int F_i \, dx_c \qquad (3)$$

---

1 Inertia force is defined as the opposite of the resultant F of all forces (internal and external) acting on the mass, so $Fd = -m \cdot \ddot{x}_c$.

For the determination of the internal force, use can be made of the deformation characteristic where the taken force equals the external force for static conditions. For the given deformation characteristic (1) it follows:

$$F_i = k \cdot x_c^n$$

This results in:

$$U_{x_c} = \frac{1}{n+1} k \cdot x_c^{n+1} \qquad (4)$$

For the deformation energy at the maximum deflection, x_c can be simply replaced by x_m which occurs at time = tm.

It is possible to address combined deformation characteristics for the total deformation characteristic of a structure: $\sum\limits_{n=0} k \cdot x_m^n$ like an elastic-plastic response or a response based on bending and membrane. The integrated deformation energies can be simply added to each other (if it is allowed for the physical behaviour):

$$U_{x_c} = \sum_{n=0} \frac{1}{n+1} k \cdot x_c^{n+1}$$

The deformation energy coincides with the area below the Fms - xm curve, either for a single deformation characteristic or the total deformation characteristic.

For many structures, the stiffness 'k' and the exponent 'n' of the deformation characteristic are not known. It is sometimes possible to determine them from equilibrium considerations. A more convenient and direct way to determine the deformation energy U is the use of the energy expression based on the stress-strain curve:

$$U = \int_{Vol} \sigma \delta \varepsilon$$

This can be worked out for the four mentioned deformation characteristics of the bending and membrane mechanism, see (Keizers, Erkel, 1992).

3.1.3 Equation of motion

The energy equation (2) cannot simply be used for the determination of a response solution. The reason is the work done: $W_e = \int F_e(t)dx_c$, where x_c is the response of the system dependent on mass and resistance function. It is evident now that the work done depends not only upon the external forces in time but also upon the properties of the system itself. Therefore the work done can only be solved directly in a very few cases (see paragraphs 3.2 to 3.4).

To get a more convenient expression, the energy equation is written as an equation of motion. For this purpose Lagrange's equation is used. So, Lagrange's equation is merely a device for writing the equation of motion rather than a method of solution.

This equation can be derived by several principles, e.g. the principle of virtual work. This means that for an equilibrium situation, the work done by the external forces during a virtual deflection δx_c must be equal to the work done of the internal[1] forces during that virtual deflection:

$$\delta We + \delta Wd = \delta Wr + \delta Wp$$

Evaluating, see (Biggs, 1964), these expressions with the relation $\delta W = \frac{dW}{dx_c} \cdot \delta x_c$ it is found for one generalized coordinate x_c that the equation of Lagrange gives:

$$\frac{d}{dt}\left(\frac{dEk}{dxc/dt}\right) - \frac{dEk}{dx_c} + \frac{dU}{dx_c} = \frac{dWe}{dx_c} \qquad (5)$$

Notice that the first term of Lagrange's equation covers all velocity dependent terms and the second term all displacement dependent terms. For all cases considered in this report, the kinetic energy consists only of velocity dependent terms, hence the second term $\frac{dEk}{dx_c} = 0$.

1 Note that for the external work done the forces on the mass are considered and for the internal work done the forces on the springs.

Applying Lagrange's equation to the expressions of the energies and work done, the equation of motion can be derived[1] for the given deformation characteristic:

$$m \cdot \ddot{x}_c + k \cdot x_c^n = F_e(t) \tag{6}$$

For the SDOF applications a similar method is often used:

The energy equation (2) can be differentiated with respect to time:

$$\frac{dW_e}{dt} = \frac{dU}{dt} + \frac{dEk}{dt} \tag{7}$$

and W_e is transformed into $\int F_e(t) \frac{dx_c}{dt} dt$.
This technique also delivers the equation of motion when it is applied to the energy expressions and $\dot{x}_c$ is cancelled (it is principally not zero).

The equation of motion (6) deals with the <u>change of energy</u> contrary to the equation of conservation of energy (2). This is very convenient while deformation energies of the total deformation characteristic which do not change, disappear, reducing the equation length. This also means that the equation of motion can be only derived for the actual deformation characteristic of a resistance mechanism (bending, membrane, etc.). This leads to a <u>set of equations of motion</u> with a change at a certain deflection criterion. Still deformation energies of different resistance mechanisms can be incorporated.

If the load-time history, Fe(t), is given, a solution of $x_c(t)$ can be determined either analytically or numerically with the equation of motion[2].

---

1 The equation of motion can also be derived from equilibrium considerations (d'Alembert), but for real structures the energy method is more powerful, convenient and consistent because the energies can be integrated over the structure.

2 Basically, the energy equation might be used as an equation of motion (based on velocity) for a numerical solution. But this is not efficient with respect to the number and nature of the terms in the equation.

In this report, only analytical solutions are considered. Complete analytical solutions only exist for the linear case where the exponent of the deformation characteristic is 1. However, there are some situations where solutions also exist for non-linear deformation characteristics. These situations are the *impulsive* and *quasi-static* loading realms or a combination. For a solution in this situation direct use is made of the general energy equation (2) instead of the equation of motion. These situations are frequently referred to in this report.

3.1.4 Dynamic Load Factor

It appears that some load time functions, e.g. shock wave, steady state vibration, the maximum dynamic deflection, can in general be expressed as the static deflection multiplied by a factor:

$$x_m = DLF \cdot x_{ms} \tag{8}$$

where x_{ms} is the static displacement as result of a static load with the magnitude of the peak of the dynamic load Fm. The ratio between the dynamic deformation and static deformation is called the Dynamic Load Factor (DLF). This factor indicates the amount of reaction (x_m) to the applied peak load (F_m).

3.1.5 Static solutions

If the dynamic load factor is known, the maximum dynamic response (x_m) is derived from the calculated static deformation as result of the peak load, multiplied with that dynamic load factor. One is usually familiar with calculating the static deflection. When the exponent of the deformation characteristic is known, the work done by the static load can be calculated:

$$W_{e_{ms}} = \int F_{ms} dx_m = \frac{1}{n+1} F_{ms} \cdot x_m$$

and according to the general energy equation (2) this should be equal to the deformation energy $U(x_m)$ because the kinetic energy is zero for static problems.

For real structures, the exponent of the deformation characteristic is not always known. To get still a static solution, many 'energy methods' are available[1]. The following convenient method is used here.
The 'total (potential) energy' is defined as:

$$U(x_m) - F_{ms} \cdot x_m$$

Notice that the last term is not the work done by the external load which has to be minimized by differentiation to x_m, and set equal to zero.

To determine the static deflection as a result of a statically applied peak dynamic load, use is made of these principles and the expression for the deformation energy (4). This results in:

$$x_{ms} = \sqrt[n]{\frac{F_m}{k}} \tag{9}$$

Combination of (1), (8) and (9) results, for the virtual static taken force:

$$F_{ms} = DLF^N \cdot F_m \tag{10}$$

It is important to notice that for a combination of deformation characteristics, a static deflection can also be derived with the mentioned principle. Applying this for the combined deformation energy (see paragraph 3.1.2) results in:

$$F_{ms} = \sum_{n=0} k_n \cdot x_m{}^n$$

The static deflection can be extracted from this equation. So, the external force for static conditions is simply the addition of the resistances of the different deformation characteristics for the static deflection.

1 Note that if the equation of motion already exists for a given problem, the static solution can be simply obtained by setting the acceleration to zero.

3.2 Impulsive loading realm

The response lies in the impulsive loading realm if the duration of the applied load is very short compared with the response time of the structure (e.g. structure loaded by the blast wave from an externally detonating HE charge), see Figure 4.

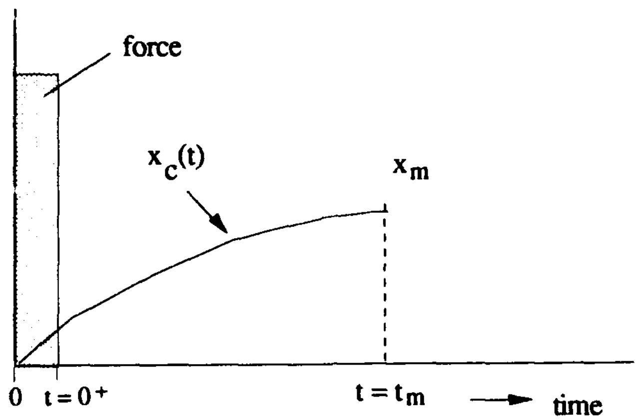

Figure 4 Impulsive loading response

For determining the maximum dynamic deflection, x_m, the conservation of energy equation (2) will be used.

The deformation is small (with respect to the maximum deformation) when the loading has ended. This implies that the deformation energy at that time (i.e. $t=0^+$) is negligible, i.e. U = 0, hence $W_{e_{t=0^+}} = E_{k_{t=0^+}}$. This means that W_e has attained a constant value when U=0 and remains constant throughout the whole period because $W_e = \int F_e \, dx$. This yields: W_e = constant:

$$U + E_k = \text{constant} \tag{11}$$

This equation is a special shape of the general energy equation (2) and it also yields for non-conservative forces, but it is only valid if no external work (W_e) is done.

When U only consists of conservative forces, Fp, the equation is known as the equation of conservation of mechanical (kinetic and potential) energy.

At the moment of maximum (t_m) deformation $x_c = x_m$ where $E_k = 0$ it follows:

$$U_{t=t_m} = E_{k_{t=0^+}} \tag{12}$$

The deformation energy is given in paragraph 3.1.2.

The kinetic energy increase for the free body motion in the impulsive loading is, according to the energy equation (2):

$$\int F_e(t)dx_c = \frac{1}{2}m\left[\frac{dx_c}{dt}\right]^2$$

which has to be worked out.

However, it is easier to use the overall Newton's second law for the determination of the maximum kinetic energy given the load-time history.

$$F = \frac{dM}{dt} \Rightarrow \Delta(m\ v_m) = \int F(t).dt \quad \text{(where } M = \text{Momentum)}$$

which is the impulse balance of the free body motion of the mass.

Impulse of pulse load is defined as I, it follows for the maximum velocity v_m and starting condition $v = 0$:

$$I = \int F(t).dt \rightarrow v_m = \frac{I}{m}$$

From the expression for the kinetic energy it follows:

$$E_{k_{t=0^+}} = \frac{I^2}{2m} \tag{13}$$

3.3 Quasi-static loading realm

If the time constant of the applied load (θ) is very long compared with response time, the loading may be considered as quasi-static (which is the opposite of the impulse load). This means that when x_m has been reached, the load has not changed significantly (e.g. nuclear explosions at long distances), see Figure 5.

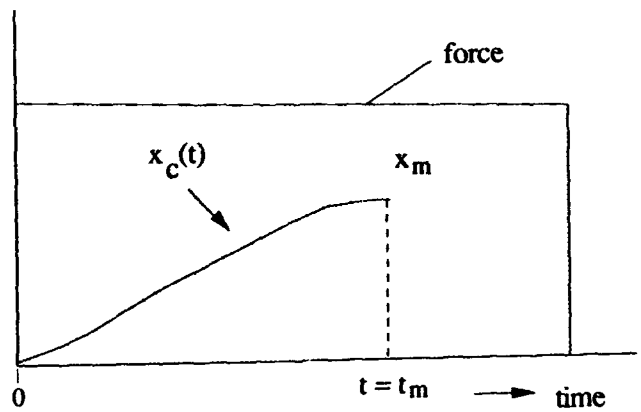

Figure 5 Quasi-static loading response

Only in this case can the work done by the applied load (W_e) be solved because F_m is independent of x_c[1]:

$$W_{e_{t=t_m}} = \int_0^{x_m} F_m \, dx_c = F_m x_m \tag{14}$$

To determine x_m (E_k=0) the expression for the quasi-static loading realm is derived from the general energy equation (2):

$$W_{e_{t=t_m}} = U_{t=t_m} \tag{15}$$

3.4 Combined loading realm

In some cases, one can distinguish an impulse and a quasi-static load in an external load time history, e.g. the loading on a heavy concrete wall during an internal detonation.

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1 When the external force F_e does not change in time and is of a conservative nature, this force can be considered as a conservative field force. This means that in fact also for a quasi-static loading the general conservation of energy equation (2) can be directly written as the equation of conservation of mechanical energy (11) while the work done by the conservative field force can be expressed as a change in potential energy.

Again from the general equation of energy (2), the maximum deflection x_m can be calculated. The work done consists of the work done by the quasi-static force and the work done by the impulse load, the latter is the initial kinetic energy. The energy relation becomes:

$$F_m x_m + \frac{I^2}{2m} = U_{t=t_m} \tag{16}$$

This expression is not applied further for any deformation characteristic because its evaluation is very similar with the other loading realms.

3.5 Transformation factors

For the application of all derived formulae in this report, neither the entire mass, stiffness nor force of a real structure can be used. These values must be multiplied with so-called transformation factors which depend on the deflection shape of the beam or plate and on the distribution of the external pressure. This can be imagined clearly if for instance the mass of the beam is considered. A part near the centre will be subjected to a higher velocity than a part near the supports.
Although the main purpose of the formulae in this report is to present formulae for a parameter study, the transformation factors will be mentioned for reasons of completeness.

The transformation factors are defined as follows:

Load factor:	Tf	=	F/Fbeam	where Fbeam is the real force on the beam
Stiffness factor:	Tk	=	k/kbeam	where Fbeam = kbeam.x_m^n for static load Fbeam
Mass factor:	Tm	=	m/mbeam	where mbeam = ρ•l•w•t*

where w = width
l = length
t* = equivalent thickness is volume per square metre
ρ = density of material

The factors can be determined by deriving the energy terms for the real structure and comparing them with the energy expressions of the one-spring-mass system.

The transformation factors for a one-way or beam[1] structure can be determined, where the deflection and pressure shape are defined as follows:

$$x = xc \cdot fx(u)$$
$$p = pc \cdot fp(u)$$

where: fx(u) = the deflection shape as a function of distance coordinate
u = distance coordinate defined along the length of the beam
fp(u) = pressure distribution as a function of distance coordinate
pc = peak pressure of distribution (mostly at centre deflection)

It follows now:

$$Fbeam = w \cdot \int_0^1 Pc \cdot fp(u) \cdot du$$

The deformation and kinetic energies, and work done per unit width are determined for the beam and compared with those for the one-spring-mass system in Chapter 3.

Work done by external forces

$$W_e/w = \int F_e/w$$

Work done on small part du of the beam is:

$$\frac{W_e}{w}(du) = \int P\,du\,dx = \int Pc\,fp(u)\,du\,fx(u)\,dx_c$$

---

1 Beam in this context does not mean only bending action but structures with a deflection shape in one direction only.

Work done on the whole beam:

$$\frac{W_e}{w} = \int \int_0^1 Pc\ fp(u) \cdot fx(u)\ dx_c\ du =$$

$$\int_0^1 fp(u) \cdot fx(u) \cdot du \cdot \int Pc\ dx_c$$

The work done of the one-spring-mass system is:

$$\int F_e\ dx_c$$

which must be equal to the work done of the beam.

With $F_e = Tf \cdot Fbeam$ it follows:

$$\int Tf \cdot w \int_0^1 Pc\ fp(u)\ du\ dx_c = w \int_0^1 fp(u) \cdot fx(u)\ du \int Pc\ dx_c$$

Tf is fully determined now.

Kinetic energy

$$\frac{Ek}{w} = \int Fd/w\ dx = \int \frac{m}{w} \frac{d}{dt} \left(\frac{dx}{dt}\right) dx =$$

$$\int \int_0^1 t^* \cdot \rho\ du \frac{d}{dt}\left(\frac{dx}{dt}\right) dx = t^* \cdot \rho \int \int_0^1 fx^2(u)\ du\ \frac{dx_c}{dt}\ d\left(\frac{dx_c}{dt}\right) =$$

$$t^* \cdot \rho\ \frac{1}{2} \left(\frac{dx_c}{dt}\right)^2 \cdot \int_0^1 fx^2(u)\ du$$

The kinetic energy of the one-spring-mass system is $\frac{1}{2}m\left(\frac{dx_c}{dt}\right)^2$ which should be equal to that of the beam. With Tf•mbeam = m it follows:

$$\frac{1}{2}\ Tm \cdot \rho \cdot t^* \cdot l \cdot w\left(\frac{dx_c}{dt}\right)^2 = w \cdot t^* \cdot \rho\ \frac{1}{2} \left(\frac{dx_c}{dt}\right)^2 \int_0^1 fx^2(u)\ du$$

Tm is fully determined now.

Deformation energy

The deformation energy of the one-spring-mass system for the maximum deflection is (see paragraph 3.1.2):

$$U = \frac{1}{N+1} k x_m^{N+1}$$

With k = Tk•kbeam and Fbeam = kbeam x_m^N it follows for the deformation energy of the statically loaded beam

$$U = \frac{1}{N+1} Tk\, w \int_0^1 Pc\, fp(u)\, du\, x_m$$

which should be equal to the static work done on the beam, see paragraph 3.1.5.
With

$$Pc = \frac{kbeam}{w \int_0^1 fp(u)\, du} x_m^N$$

it follows for the static work done on the beam:

$$W_{e_{ms}} = \frac{1}{N+1} Pc\, x_m\, w \int_0^1 fp(u) \cdot fx(u)\, du$$

Evaluating these expressions it appears that Tk = Tf. In fact this is not surprising given the fact that $x_m^N = \frac{F}{K} = \frac{Fbeam}{Kbeam}$.

It is important to notice that for a combination of deformation characteristics, the definition for the stiffness factor does not change. The expression for the static beam force becomes then: $Fbeam = \sum_{n=0} k_n \cdot x_1^{\,n}$ and all stiffnesses k_n have to be multiplied by the stiffness factor Tk. This statement agrees with the statically taken force, see paragraph 3.1.5.

Note, however, that the change of the deflection shape may not have the same influence on the change of the stiffness kn for the different deformation characteristics.
If the factors for one-way structures are summarized, they yield:

$$\text{Load factor } Tf = \text{Stiffness factor } Tk = \frac{\int_0^1 fp(u)\cdot fx(u)\,du}{\int_0^1 fp(u)\,du} \tag{17}$$

$$\text{Mass factor } Tm = \frac{1}{1}\int_0^1 fx^2(u)\,du$$

It can be shown that all deflection expressions for the quasi-static loading realm can be directly applied because the Load factor equals the Stiffness factor, which is very convenient. This is not the case for the impulsive loading realm or for all other response formulae where the transformation factors have to be input.

For the determination of Vm (see page 21) also the transformation factors should be used for the impulse equation:

$$\dot{x}_{t=o^+} = Vm = \frac{Tk\int Fbeamdt}{Tm \;\; mbeam}$$

And the kinetic energy becomes (equation 13):

$$Ek_{t=o^+} = \frac{1}{2}\cdot Tm \;\; m\ beam \cdot \dot{x}^{\,2}_{c_{t=o^+}}$$

It should be kept in mind however that the calculated velocity at the end of the impulsive loading realm do not well match reality. This is caused by the fact that just in a very early stage of the response the deflection shape deviates largely from the asssumed statically deflection shape. The user should be aware of this possible mismatch which leads amongst others to a difference between the impulse of the shockwave and the momentum of the beam.

If the expressions for the work done, the deformation energy and the kinetic energy are input in the Lagrange's equation, an equation of motion arises for the whole beam. This equation can be rewritten as:

$$\frac{Tm}{Tk}\, mbeam \cdot \ddot{x}_C + kbeam \cdot x_C{}^{n} = Fbeam(t)$$

So, for the equations of motion, only one transformation factor is needed: the Mass factor Tm divided by the Stiffness factor Tk.
In (Biggs, 1964), a set of transformation factors is worked out and presented for only uniformly distributed loads.

4 SOLUTIONS FOR LINEAR DEFORMATION CHARACTERISTIC

The linear deformation characteristic is very important. It forms the basis for many elastic calculations.
The well-known equation of motion for this type of structure follows from equation (6) with n = 1:

$$m\ddot{x}_C + kx_C = F_e(t)$$

which will be solved for a load-time history $F_e(t)$ from a shock wave load.

4.1 Taylor wave loading

The pressure time profile of a free field or incident shock wave in water and air can be rather well expressed by the Taylor shape which has the form (see Figure 6):

$$p(t) = P_f \cdot e^{-t/\theta}$$

where: P_f is the incident peak pressure, and

θ is time constant of the applied load (Taylor shape).

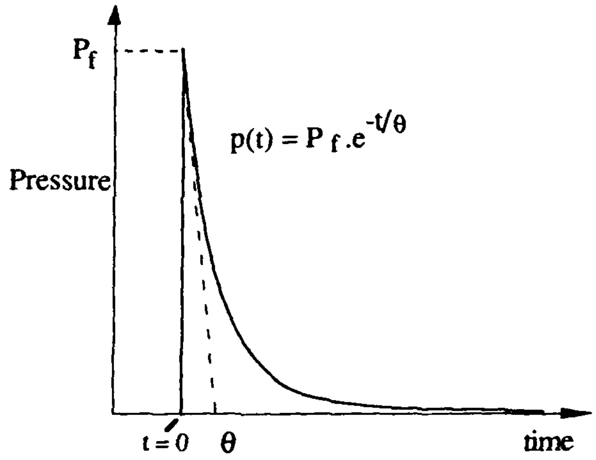

Figure 6 The Taylor shock wave

For a one-dimensional reflection in air, the loading on the structure has exactly the same time profile; only the peak pressure is increased with a so-called reflection factor RF. The resulting loading on the structure is now:

$$F_e(t) = F_m \cdot e^{-t/\theta} \tag{18}$$

where F_m is the peak force = $P_f \cdot A \cdot RF$ (A = surface).

The peak pressure on the structure: $P_m = \frac{F_m}{A} = P_f \cdot RF$

The impulse of this force is:

$$I = \int F(t)\,dt = F_m \cdot \theta \tag{19}$$

The specific impulse on the structure is: $I_s = \int P(t)\,dt = P_m \cdot \theta = \frac{I}{A}$

If θ is long compared to the structural response time, the quasi-static loading realm results. If θ is short, the impulsive loading realm results.

4.2 Complete loading realm (from impulsive until quasi-static)

The equation of motion can be solved as follows. The homogeneous part of the solution can be written as: $x_c(t) = C1.\cos\omega t + C2.\sin\omega t$. When this solution is put in the equation of motion for a free vibration it follows:

ω - natural circular frequency $= \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$ [rad/s] $\quad (f = \frac{\omega}{2\pi})$

T - natural vibration period [s]

Assuming that the particular solution can be described by $x_c(t) = C3.e^{-t/\theta}$, and inputting this in the forced equation of motion, yields: $C3 = \dfrac{F_m}{\frac{m}{\theta^2} + k}$

Finally, the general solution is composed of the two parts and for starting conditions, $x_c = 0$ and $dx_c/dt = 0$, it can be written:

$$X_c(t) = \frac{F_m}{K} \cdot \frac{1}{1 + \frac{1}{(\omega \cdot \theta)^2}} \left(\frac{\sin\omega \cdot t}{\omega \cdot \theta} - \cos\omega \cdot t + e^{-\frac{\omega \cdot t}{\omega \cdot \theta}} \right) \qquad (20)$$

The results of this equation are depicted in Figure 7 where for $\frac{x_c(t)}{F_m/k}$ the $\omega\theta$ influence can be clearly detected.

Note that the time of maximum response will increase for increasing values of $\omega\theta$.

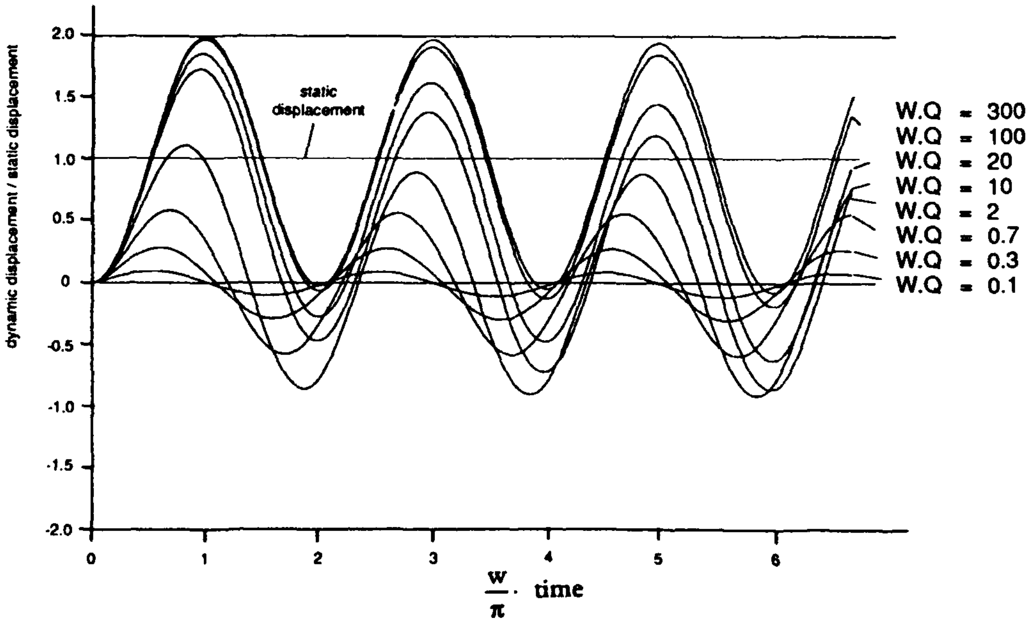

Figure 7 Comparison of the influence of the $\omega \cdot \theta$ factor on the response of a linear spring mass system loaded by a shock wave with a constant peak force

xc(t) gets a maximum at tm as $dx_c(t)/dt = 0$, which yields:

$$\cos \omega \cdot tm + \omega \cdot \theta \cdot \sin(\omega \cdot tm) - e^{-\frac{\omega \cdot tm}{\omega \cdot \theta}} = 0 \qquad (21)$$

The parameter ω.tm can be established from this relation for a given ωθ value, which is regretfully of an implicit nature. Putting the ω.tm values, which can be numerically found, and the according ωθ value into the solution for xc(t), the maximum relative deflection, $\frac{xm}{Fm/k}$ can be found.

In accordance with paragraph 3.1.5, this yields, for the static deflection of the linear deformation characteristic as a result of the statically applied peak load Fm:

$$x_{ms} = \frac{Fm}{k}$$

This means that in accordance with the Dynamic Load Factor definition (paragraph 3.1.4)

$$DLF = \frac{x_m}{Fm/k_{be}} \quad \text{for the linear case} \qquad (22)$$

It appears that from equations (21) and (20), the DLF can be solved if only the ωθ value is known, see Figure 8.

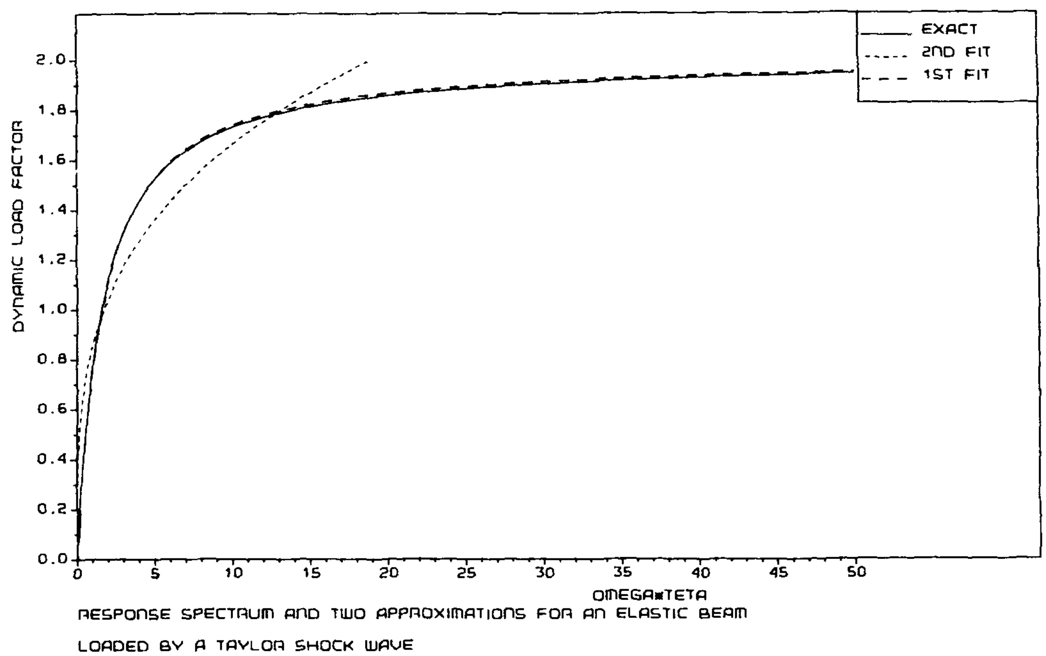

Figure 8 Response spectrum and two approximations for an elastic beam loaded by a Taylor shock wave

In general, it can be stated that for the linear deformation characteristic, the DLF depends only upon the natural frequency ω of the structure and on a characteristic time of a certain loading type. For the Taylor wave this is the time constant θ. A general plot of the DLF is schematically given in Figure 9. This important curve is only valid for a linear elastic deformation characteristic.

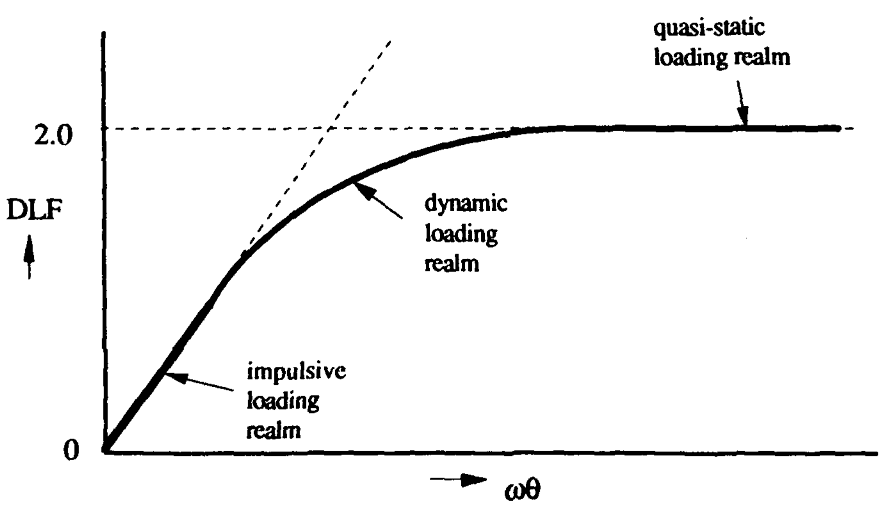

Figure 9 Dynamic response curve for linear deformation characteristic

In (Biggs, 1964), DLF-like curves for the bending mechanism (elastic and plastic) are given for four types of loadings other than the Taylor wave, including the time of maximum response.

DLF can be solved explicitly only for two specific cases. These two cases, the ***impulsive*** and ***quasi-static*** loading realms, respectively, have already been mentioned in Chapter 3 and are depicted in Figure 9. For determining the maximum deflections x_m and the corresponding DLFs, the analytical solutions (20) and (21) can be used for the linear deformation characteristic only, but to show the principle, the derived energy methods in Chapter 3 will be used.

4.3 Impulsive loading realm

From the $x_c(t)$ solution (20) it appears that for a impulsive loading realm: $x_c = x_m \sin\omega t$, see Figure 10.

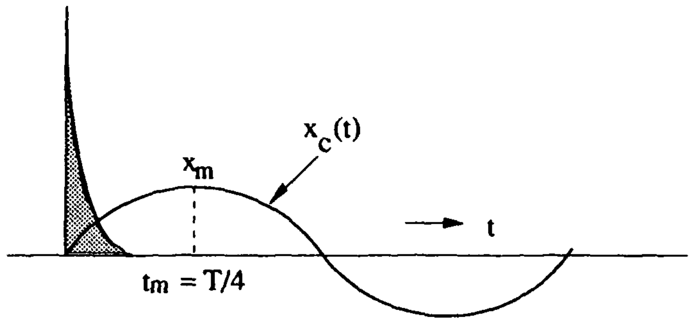

Figure 10 Impulsive loading response for linear case

For the deformation energy, use is made of expression (4) for the maximum response:

$$U_{t=t_m} = \frac{1}{2} k_{be}\, x_m^2$$

The kinetic energy can be derived from formula (13) and the impulse of the Taylor wave (19):

$$v_m = \frac{F_m \cdot \theta}{m} \rightarrow E_{k_{t=0^+}} = \frac{[F_m \cdot \theta]^2}{2m}$$

For determining the maximum dynamic deflection x_m, the derived principle (12) is used. Substituting the above-mentioned results, leads to:

$$x_m = \frac{I}{\sqrt{mk_{be}}} \quad \text{(impulsive loading realm)} \tag{23}$$

Note that in contrast to the static loading, x_m is proportional to $\frac{1}{\sqrt{k_{be}}}$ instead of $\frac{1}{k_{be}}$.
Using $\omega^2 = \frac{k_{be}}{m}$ and $I = M = F_m \cdot \theta$, x_m can also be written as:

$$x_m = \frac{v_m}{\omega}$$

(which agrees with the time-differentiated time shape of the response) or in the DLF shape with respect to the static deflection x_{ms}:

$$x_m = \omega\, \theta \frac{F_m}{k_{be}} \tag{24}$$

Apparently, in comparison with equation (22)

$$DLF = \omega\theta. \tag{25}$$

This realm is the left oblique asymptote of Figure 9 for a Taylor load and will be used from $0 \le \omega\theta \le 0.30$, where the maximum error in DLF is +4.3%.

4.4 Quasi-static loading realm

From the $x_c(t)$ solution (20), it appears that $x_c = x_m \frac{1}{2}(1 - \cos\omega t)$, see Figure 11.

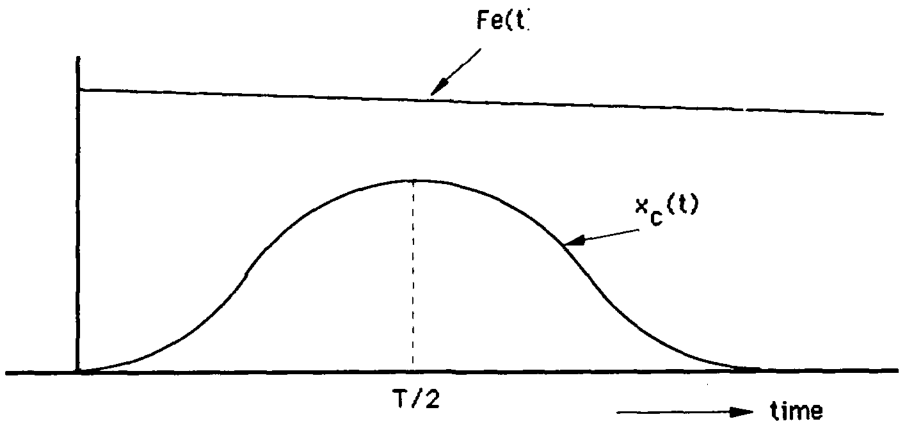

Figure 11 Quasi-static loading response for linear case

Substitution of the corresponding expressions into the derived principle (15) yields:

$$F_m x_m = \frac{1}{2} k_{be} x_m^2 \rightarrow x_m = 2 \frac{F_m}{k_{be}}$$

Apparently, the dynamic load is twice the applied static load, or, in comparison with equation (22), the dynamic load factor

$$DLF = 2.0 \tag{26}$$

This realm is the horizontal asymptote in Figure 9 and will be used for the Taylor load for $\omega\theta \geq 30$, where the maximum error in DLF is +5.2% and is usually called the quasi-static loading realm.

4.5 Application of transformation factors

To show the usefulness of the transformation factors, they are applied for a case of the linear deformation characteristic.

A beam is considered loaded by a sinusoidal pressure distribution along its length and a sinusoidal deflection shape:

$$fx(u) = fp(u) = \sin\frac{\pi.u}{l}$$

By using the mentioned expressions for the transformation factors it can be found:

Load factor Tf = Stiffness factor Tk = π/4

Mass factor Tm = 1/2

These constants have to be multiplied with the entities of the beam before using them in the formulae presented in this report. The impulsive loading realm for the linear deformation characteristic is considered:

$$x_m = \omega\,\theta\frac{F_m}{k_{be}}\,\frac{Tf}{Tk}$$

It seems that x_m can be directly calculated because $\frac{Tf}{Tk} = 1$, but the natural frequency has also to be considered.

$$\omega = \sqrt{\frac{k_{be}}{m}} = \sqrt{\frac{Tk \cdot kbeam}{Tm \cdot mbeam}}$$

For a simple supported beam under sinusoidal deflection shape, the stiffness can be derived by determining the bending deformation energy given the deflection shape and using the techniques for static response. The stiffness is:

$$kbeam = \left(\frac{\pi}{l}\right)^3 \cdot 2 \cdot EI$$

When all the expressions are substituted in the natural frequency equation, they result in:

$$\omega beam = \left(\frac{\pi}{l}\right)^2 \sqrt{\frac{EI}{\rho \cdot w \cdot t^*}}$$

which is exactly identical to the classic value for the first natural frequency of a simple supported beam. This is not surprising because the chosen deflection shape and load distribution are comparable with the conditions of a beam in the first natural vibration.

4.6 Approximation of DLF curve

Because the linear elastic deformation characteristic is commonly used for design purposes, an explicit function for the whole DLF-$\omega\theta$ range is desired.
The following convenient approximation has been found:

$$DLF = \frac{a}{\left[1 + \frac{b}{(\omega\theta)^c}\right]} \tag{27}$$

For the Taylor wave, numerical approximation yields: a = 2.0 , b = 1.68 and c = 1.04 .
The accuracy of this approximation is excellent, see Figure 8. When $\omega\theta$ is small, then the equation DLF = $\omega\theta$ evolves, and when $\omega\theta$ is large, then DLF=2.0 evolves, which are both in agreement with the theoretical impulsive and quasi-static loading realms, respectively.

For some applications such as deriving scaling laws, further simplifications are needed. The Taylor DLF-$\omega\theta$ relation was then approximated by the following relation:

$$DLF = a \ (\omega\theta)^b$$

(Note that a = 1, b = 1 for the impulsive loading realm and that a = 2.0, b = 0 for the quasi-static loading realm.)

As a further simplification, the response curve has been approximated roughly between the two asymptotes: $0.5 \le \omega\theta \le 20$. From a least square approximation, the constants a and b were found as:

$$a = 0.85 \quad \text{and} \quad b = 0.29$$

This approximation turned out to be inferior, see Figure 8.

However, with respect to the scaling of one shock wave to another, a different approximation can be used. Scaling in this context means to get an equal maximum dynamic deflection x_m of an identical structure for two Taylor shock wave loadings. Basically,

$$x_m = \text{constant} = DLF\frac{F_m}{kbe}$$

So, the ratio $\frac{x_{m_2}}{x_{m_1}} = 1$, or $DLF_2/DLF_1 = F_{m_1}/F_{m_2}$, where F_{m_1}/F_{m_2} is constant (independent from $\omega\theta$). This results in the requirement:

$$\left[\frac{DLF_2}{DLF_1}\right]_{exact} \approx \left[\frac{DLF_2}{DLF_1}\right]_{approx} = \left[\frac{\omega\theta_2}{\omega\theta_1}\right]^b$$

where b minimizes the sum of all the relative errors, given by:

$$E_{rel} = \sum_{i;j=1}^{N}\left[\frac{\frac{DLF_i}{DLF_j} - \left[\frac{\omega\theta_i}{\omega\theta_j}\right]^b}{\frac{DLF_i}{DLF_j}}\right]$$

Computations showed that the minimum error E_{rel} occurred for $b = 0.32$.
Substitution of some values of $\omega\theta$ in the above-found approximation indicated that this approximation was still not satisfactory. The errors found were in excess of 30%.
However, when a piece-wise approximation was used for the DLF–$\omega\theta$ curve based on this technique, the errors were reduced significantly. The curve was approximated in four parts, which are listed in Table I, including the values for b that were found by the numerical method.

Table I Piece-wise approximation of DLF ratio curve

Region	$\omega\theta$-range	b	Type of loading	max. error [%]	mean error [%]
I	0 - 0.55	0.948	approx. impulsive loading	5.3	1.8
II	0.55 - 2.0	0.632	dynamic loading (1st part)	6.0	2.1
III	2.0 - 10.0	0.270	dynamic loading (2nd part)	6.7	2.6
IV	10.0 - 200	0.045	approx. quasi-static loading	4.5	1.8

It must be noted that for an arbitrary load applied to the DLF–$\omega\theta$ curve in regions I and IV, approximately the same value for b as for the Taylor load is found. The other regions, however, have to be redefined with respect to $\omega\theta$ resulting in different values for b.

A plot of region II of the curve is depicted in Figure 12.

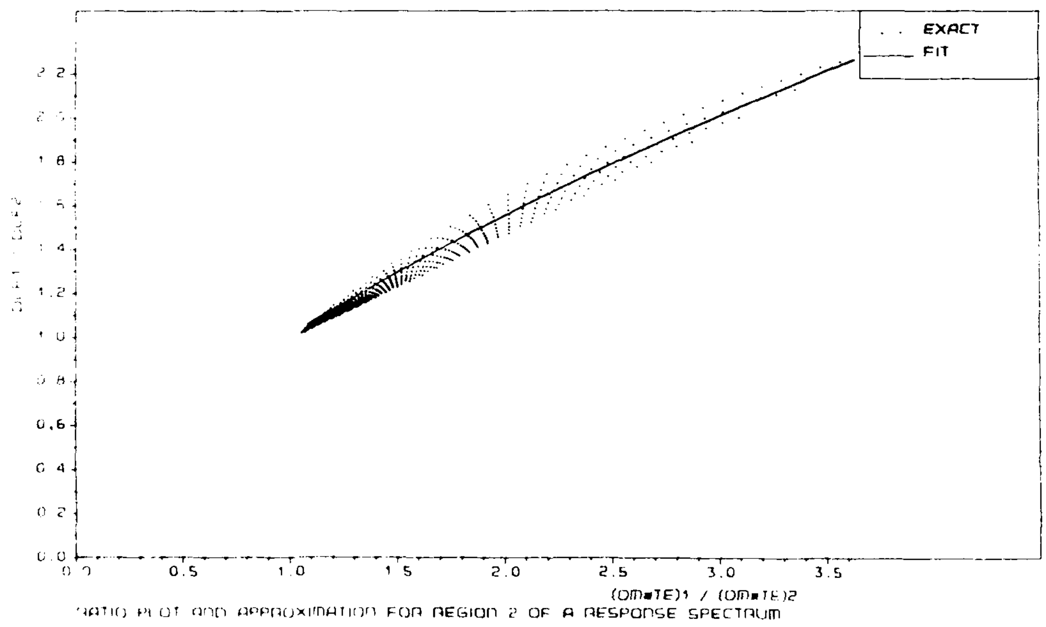

Figure 12 Ratio plot and approximation for region II of a response spectrum

5 SOLUTIONS FOR NON-LINEAR DEFORMATION CHARACTERISTICS

For non-linear deformation characteristics ($n>1$), the equation of motion (6) cannot be solved analytically for the Taylor load.

Therefore the maximum deflection x_m or the DLF cannot be found in the whole range from the impulsive loading to the quasi-static loading realms. Due to the non-linearity in the deformation characteristic, the peak load F_m appears in the response relation in a different way for the different realms. Also it is not possible to determine a natural frequency while the time of maximum response depends on the load.

However, solutions for the mentioned deformation characteristics (general, bending, membrane and combination) can be found for the impulsive and the quasi-static loading realm with the aid of the derived principles (12) and (15), respectively.

5.1 Arbitrary deformation characteristic

For all single deformation characteristics in the two loading realms, an expression can be found which has the shape of a DLF times the static deflection as result of the peak load.

A general function for this DLF for an arbitrary value of the exponent (n) of the deformation characteristic can be simply derived using the general deformation characteristic equation (1) and the general expression for the DLF equation (8) in combination with static resulting deflection according to (9).

5.1.1 Impulsive loading realm

Applying the impulsive loading realm principle (12) yields:

$$\frac{1}{n+1} k \cdot x_m^{\,n+1} = \frac{I^2}{2m} \rightarrow$$

$$x_m = I^{\frac{2}{n+1}} \left(\frac{n+1}{2 \cdot m \cdot k}\right)^{\frac{1}{n+1}} \tag{28}$$

and

$$DLF = \left(\frac{k}{F_m}\right)^{\frac{1}{n^2+n}} \cdot \left(\frac{I^2}{F_m}\,\frac{n+1}{2 \cdot m}\right)^{\frac{1}{n+1}}$$

In the impulsive loading realm, the maximum deflection x_m is only governed by the impulse of the shock wave. This is important, because the maximum deflection is <u>independent</u> of the peak pressure, decay constant and the shape of the shock wave; only the impulse of the shock wave determines the loading (from this impulse the maximum velocity or kick-off velocity, thus kinetic energy, of the structure is directly related). The impulse does not necessarily originate from a shock wave. Every pressure-time profile with equal impulse delivers the same deflection in this realm.
The impulse loading, I, from a general reflected shock wave can be expressed as

$$I = F_m \cdot t_p \cdot s \tag{29}$$

The parameter t_p is the phase duration of the shock wave and the factor s relates to the shape of the shock wave. For the Taylor wave, $t_p = \theta$ and $s = 1$. Given this equation, a more general expression for the DLF can be found:

$$DLF = \left(\frac{k}{F_m}\right)^{\frac{1}{n^2+n}} \cdot \left(F_m \cdot t_p{}^2 \cdot s^2\,\frac{n+1}{2 \cdot m}\right)^{\frac{1}{n+1}} \tag{30}$$

As can be seen, only for a linear case, i.e. n=1, does the peak load disappear from the relation which agrees with the relation derived earlier for the linear case.

5.1.2 Quasi-static loading realm

Applying the principle (15) for the quasi-static loading results in:

$$\frac{1}{n+1} k \cdot x_m{}^{n+1} = F_m\, x_m \quad \rightarrow$$

$$x_m = \sqrt[n]{n+1} \cdot \sqrt[n]{\frac{F_m}{k}} \tag{31}$$

This yields:

$$DLF = \sqrt[n]{n+1} \tag{32}$$

So, DLF can vary between 1.0 and 2.72 for a quasi-static loading.

5.2 Plastic deformation characteristic

5.2.1 Impulsive loading realm

First, the plastic deformation characteristic is analysed separately from the elastic prehistory which is justified when the plastic deformation dominates.

Only the impulsive loading realm is considered for the plastic deformation characteristic of the bending mechanism because a quasi-static load would result in infinite displacements (n=0) if the applied load exceeds the plastic resistance force (F_{bp}). On the other hand, no deformation will occur if the quasi-static load is smaller than F_{bp}.

For the impulsive loading realm, the internal work done can be written as:

$$U_{t=t_m} = F_{bp} \cdot x_m$$

Applying the impulsive loading realm principle (12) yields:

$$x_m = \frac{I^2}{2m \cdot F_{bp}} \qquad (F_m > F_{bp}) \tag{33}$$

It can be written: $x_m = \text{constant}.I^2$

In fact, if $F_m > F_{bp}$, the equation of motion becomes the free body motion for the whole loading realm with:

$$\ddot{x}(t) = \frac{F(t) - F_{bp}}{m}$$

and

$$x(t) = \int \dot{x}(t)\,dt = \frac{F_m \cdot \theta}{m} \cdot (t + \theta \cdot e^{-t/\theta} - \theta) - \frac{F_{bp}}{2.m} \cdot t^2$$

where the maximum deflection occurs at time where $\dot{x}(t) = 0$ which follows from the criterion

$$\frac{F_{bp}}{F_m} = \frac{1 - e^{-\alpha}}{\alpha} \quad \text{and where t can be determined from } t = \alpha.\theta$$

5.3 Elastic membrane deformation characteristic

The plastic non-linear deformation characteristic can be regarded as a plastic string and it is comparable with the linear elastic deformation characteristic. Also a kind of natural frequency can be defined.

For the impulsive and quasi-static loading realms of the elastic membrane mechanism, n=3, an expression for x_m can be found, using the relations for the arbitrary deformation characteristic.

5.3.1 Impulsive loading realm

Putting n = 3 into equation (28) leads to:

$$x_m = 1.19 \sqrt{\frac{F_m\,\theta}{\sqrt{k_{me} \cdot m}}} \qquad (1.19 = \sqrt[4]{2}) \tag{34}$$

Hence, for the impulsive loading realm: $x_m = \text{constant}\ \sqrt{I}$

Using equation (29) and (30) yields, for the DLF:

$$DLF = \sqrt{I}\left(\frac{k_{me}}{F_m}\right)^{0.1} \cdot \left(\frac{2}{m \cdot F_m}\right)^{0.25} \qquad (35)$$

5.3.2 Quasi-static loading realm

For the quasi-static loading realm, the application of (31) yields:

$$x_m = 1.59\left[\frac{F_m}{k_{me}}\right]^{1/3} \qquad (1.59 = \sqrt[3]{4}) \qquad (36)$$

and from (31):

$$DLF = 1.59$$

5.4 Combined deformation characteristics

For combined deformation characteristics, the total deformation characteristic is composed of several single deformation characteristics: $\sum_{n=0} k \cdot x_c^{\,n}$ see paragraph 3.1.2.

5.4.1 Deformation energies

Until now, the plastic part of the deformation characteristic has been dealt with separately from the elastic part. In practice, the plastic part is preceded by an elastic one.

For the bending beam, the elastic phase is followed by the formation of plastic hinges which finally yields to a deformation characteristic with a constant retarding force (strain hardening is neglected for the dynamic situation). To derive the combined elastic-plastic response, the terms for the deformation energy U, for plastic deflections, can be simply added:

$$U_{t=t_m} = F_{bp} \cdot (x_m - \frac{1}{2} x_{be}) \qquad \text{(Bending)} \qquad (37)$$

It follows for an elastic-plastic response for the membrane mechanism:

$$U_{t=t_m} = \frac{F_{memax}}{2 \cdot x_{me}} x_m^2 - \frac{1}{4} F_{memax} \cdot x_{me} \qquad \text{(Membrane)} \tag{38}$$

Even the bending and membrane action can occur together. For instance, large deflections in a stiffened panel induce membrane forces in the plate while the stiffeners can still act as bending elements. In such a case, the two deformation energy terms (bending and membrane) have to be added and the maximum deflection x_m can be solved analytically. It should be kept in mind that the value for the plastic bending force is only based on the stiffeners. So, F_{bp} reduces with respect to the bending mechanism only.

For large plastic deformations, the plastic energy dominates and the elastic can be neglected, which yields, for the combined bending and membrane action:

$$U_{t=t_m} = \frac{F_{memax}}{2 \cdot x_{me}} x_m^2 + F_{bp} \cdot x_m \qquad \text{(Bending and Membrane)} \tag{39}$$

5.4.2 Impulsive loading realm

For the impulsive loading realm, it follows, for the maximum deflection of the elastic plastic bending mechanism by equating the deformation energy with the kinetic energy (12):

$$x_m = \frac{I^2}{2.m \cdot F_{bp}} + \frac{1}{2} x_{be} \qquad \text{if } I > x_{be} \cdot \sqrt{k_{be} \cdot m} \tag{40}$$

and for the elastic plastic membrane mechanism:

$$x_m = \sqrt{\frac{I^2 \cdot x_{me}}{m \cdot F_{memax}} + \frac{1}{2} x_{me}^2} \qquad \text{if } I > \frac{1}{2}\sqrt{2}\, x_{me}^2 \cdot \sqrt{k_{me} \cdot m} \tag{41}$$

The impulsive loading realm for the combined plastic bending and plastic membrane mechanism reads:

$$x_m = \sqrt{\frac{F_{bp}^2 \cdot x_{me}^2}{F_{memax}^2} + \frac{I^2}{m}\frac{x_{me}}{F_{memax}}} - \frac{F_{bp}}{F_{memax}} x_{me} \tag{42}$$

5.4.3 Quasi-static loading realm

The solutions for the maximum dynamic deflection can be calculated with the aid of the energy principle expressed in equation (15).

For the elastic-plastic bending mechanism, it follows:

$$x_m = \frac{x_{be}}{2} \cdot \frac{1}{1 - \frac{F_m}{F_{bp}}} \qquad \text{if } \frac{1}{2}F_{bp} < F_m < F_{bp} \tag{43}$$

Note that for $F_m = \frac{1}{2} F_{bp}$, it follows $x_m = x_{be}$ which agrees with a DLF = 2.

For the elastic plastic membrane mechanism, this results in:

$$x_m = \frac{x_{me}}{F_{memax}} \cdot \left(F_m + \sqrt{F_m^2 + \frac{1}{2} F_{memax}^2}\right) \quad \text{if } F_m > \frac{F_{memax}}{4} \tag{44}$$

For the quasi-static loading realm of the combined plastic bending membrane mecnanism, this leads to:

$$x_m = \frac{2.x_{me}}{F_{memax}} (F_m - F_{bp}) \tag{45}$$

5.4.4 Static solution

The static force, F_{ms}, can be simply determined from the minimization of the total potential energy, as was stated in paragraph 3.1.5.

$$\frac{d}{dx_m}\left(U_{t=tm} - F_{ms} \cdot x_m\right) = 0 \quad \rightarrow$$

$$F_{ms} = F_{bp} + \frac{F_{memax}}{x_{me}} \cdot x_m \tag{46}$$

5.4.5 Survey of maximum deflection expressions

The derived solutions for the elastic-plastic bending and membrane mechanism and are summarized in Table II.

Table II The maximum dynamic deflection, x_m, for two loading realms

mechanism	deformation characteristics	impulsive loading	quasi-static loading
BENDING	elastic	$x_m = \frac{I}{\sqrt{k_{be} \cdot m}}$	$x_m = 2\frac{F_m}{k_{be}}$
	elastic-plastic	$x_m = \frac{I^2}{2 \cdot m \cdot F_{bp}} + \frac{1}{2} x_{be}$	$x_m = \frac{x_{be}}{2} \cdot \frac{1}{1\frac{F_m}{F_{bp}}}$
		if $I > x_{be} \cdot \sqrt{k_{be} \cdot m}$	if $\frac{1}{2} F_{bp} < F_m < F_{bp}$
MEMBRANE	elastic	$x_m = 1.19\sqrt{\frac{I}{\sqrt{k_{me} \cdot m}}}$	$x_m = 1.59\left[\frac{F_m}{k_{me}}\right]^{1/3}$
	elastic-plastic	$x_m = \sqrt{\frac{I^2 \cdot x_{me}}{m \cdot F_{memax}} + \frac{1}{2} x_{me}^2}$	$x_m = \frac{x_{me}}{F_{memax}} \cdot \left(F_m + \sqrt{F_m^2 + \frac{1}{2} F_{memax}^2}\right)$
		if $I > \frac{1}{2}\sqrt{2}\, x_{me}^2 \cdot \sqrt{k_{me} \cdot m}$	if $F_m > \frac{F_{memax}}{4}$

6 PRESSURE-IMPULSE DIAGRAMS

6.1 General description

The DLF–ωθ curve can also be transformed into a so-called P-I diagram, see Figure 13. The difference is that the DLF–ωθ diagram shows the displacement as a function of relative time (ωθ), while the P-I diagram shows the required combinations of I(mpulse) and P(ressure) for a given deflection (valid for a typical structure), e.g. the displacement where the yield limit is reached. Curves in this P-I diagram may be called an iso-deflection or iso-damage curve, because it connects points having the same deflection. These curves are often used in relation to the damage due to explosions, therefore the two typical parameters of a reflected blast wave are used, i.e. the peak pressure P_m and the positive specific impulse I_s.

Note that for structures loaded by a uniformly distributed pressure of a reflected shock wave: $P_m = F_m/A$ and $I_s = I/A$.

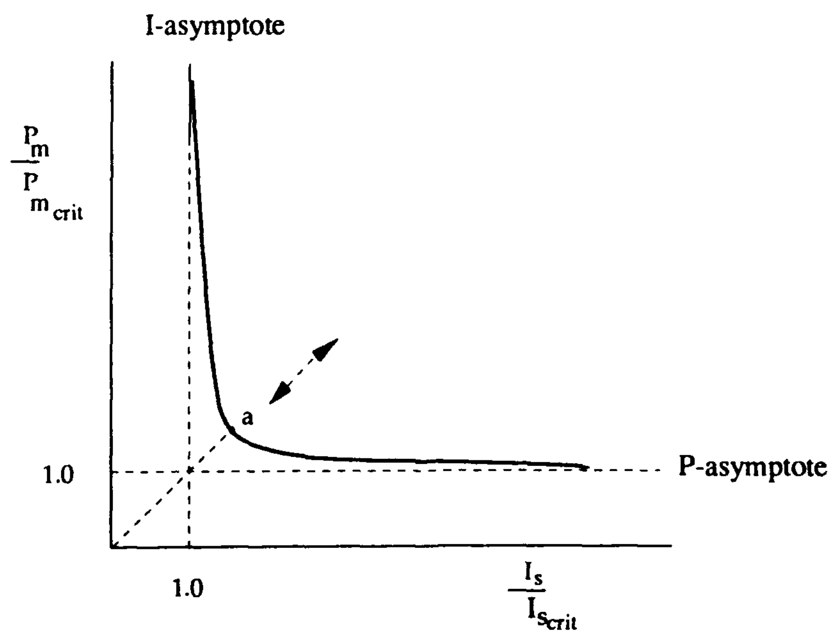

Figure 13 Iso-deflection curve in a P-I diagram

When scaling explosive charges to obtain equal displacements, this curve may be very suitable. Any point of a combination of P_m and I_s under this iso-deflection curve will stay under an allowable deflection, and any combination above the curve will lead to unallowable damage.

6.2 Approximation of an iso-deflection curve

The iso-deflection curve can be approximated by a hyperbola as:

$$\left[\frac{P_m}{P_{m_{crit}}}-1\right]\left[\frac{I_s}{I_{s_{crit}}}-1\right]=a \tag{47}$$

For the linear deformation characteristics and the Taylor wave it is known that:

$$I_{s_{crit}}=x_{mcrit}\frac{k_{be}}{\omega \cdot A} \quad \text{and} \quad P_{m_{crit}}=x_{mcrit}\frac{k_{be}}{2 \cdot A}$$

With $x_m = DLF\frac{F_m}{k_{be}}$ and $I_s = P_m \cdot \theta$ it follows:

$$\frac{I_s}{I_{s_{crit}}}=\frac{\omega\theta}{DLF} \quad \text{and} \quad \frac{P_m}{P_{m_{crit}}}=\frac{2}{DLF}$$

and x_m is a certain arbitrary centre maximum value.

For the interception of the hyperbola and the bisectrice (point 'a' in Figure 13) it yields:

$$P_m/P_{m_{crit}} = I_s/I_{s_{crit}} \rightarrow \theta = I_{s_{crit}}/P_{m_{crit}} = 2.0/\omega$$

With the $\omega\theta$ information from the point of intersection[1], the DLF can be found and the constant 'a' of equation (47) could be defined.

However, this is done by comparing all points (instead of one point from the intersection) with a numerical fit. To this end the iso-deflection curve can be transformed into a DLF–$\omega\theta$ diagram and equation (47) can be written in closed form as:

$$DLF=\frac{1+\frac{1}{2}\left\{\omega\theta-\sqrt{(\omega\theta)^2-\omega\theta(4-8a)+4}\right\}}{(1-a)} \tag{48}$$

---

1 Note that this is the same value for θ as the point of intersection in the DLF curve of Figure 9.

If this equation is approximated with respect to the actual DLF$-\omega\theta$ curve, a maximum error of only 1.8% is found for a=0.74, see Figure 14, where the exact and the approximated P-I curves are given. The hyperbola suggests symmetry around the $P_{m_{crit}}$ and $I_{s_{crit}}$ asymptotes, which is not exactly true. Also, the approximation using the hyperbola for values close to the asymptotes, is less than the explicit DLF$-\omega\theta$ curve fit suggests. This is inherent to the way the P-I diagram is constructed.

The hyperbola can be rearranged by multiplying with $P_{m_{crit}} \cdot I_{s_{crit}}$ into:

$$(P_m - P_{m_{crit}})\cdot(I_s - I_{s_{crit}}) = a\, P_{m_{crit}} \cdot I_{s_{crit}} = a\, x_{m_{crit}}{}^2 \frac{k_{be}{}^2}{2\omega.A^2}$$

Applying this to a single structure, i.e. k_{be} and ω and A are <u>equal</u>, then:

$$(P_m - P_{m_{crit}})\cdot(I_s - I_{s_{crit}}) = \text{const} \cdot x_m^2 \qquad (49)$$

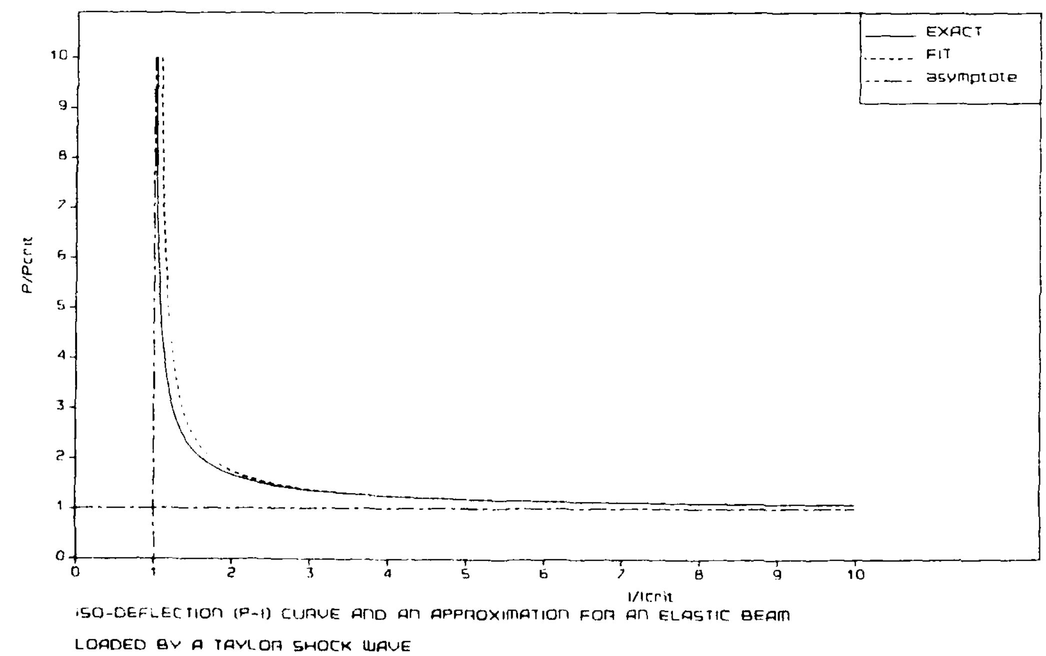

Figure 14 **Iso-deflection (P-I) curve and an approximation for an elastic beam loaded by a Taylor shock wave**

(Held, 1983) postulates that equation (47) can be simplified by neglecting $P_{m_{crit}}$ and $I_{s_{crit}}$ into:

$$P_m \cdot I_s = P_m \cdot (P_m \cdot \theta) = P_m^2 \cdot \theta$$

However, neglecting $P_{m_{crit}}$ and $I_{s_{crit}}$ can lead to unjustified use, because the last expression is valid only for limited cases, where the dynamic part of the DLF curve is approximated and the two asymptotes of the impulsive and quasi-static loading realm are neglected.

Another problem is the introduction of a so-called damage number which is equal to:

$$DN = (P_m - P_{m_{crit}}) \cdot (I_s - I_{s_{crit}}) \qquad (50)$$

where the damage is equivalent to a certain critical deflection, i.e. $DN + x_m$.
In (Held, 1983) it is suggested that when the critical deflection is doubled, the corresponding damage number will double as well, which would yield a new relation for a certain deflection. However, the statement that $DN+x_m$ is valid only when scaling for both the same structure and the same x_m, because for different x_ms $DN + x_m^2$ was derived for a linear elastic deformation characteristic, see (49).
For the other deformation characteristics, an iso-deflection curve can be derived, despite the fact that an analytical expression of $x_m = DLF.x_{ms}$ does not exist.
For a non-linear deformation characteristic, a P-I diagram can be derived only by means of numerical integration. This diagram has virtually the same shape as for the elastic deformation characteristic. Hence, as an approximation a hyperbola can be used. For the elastic membrane mechanism, the hyperbola constant 'a' = 0.6 (instead of 0.74 for the linear elastic characteristic). This curve is valid for all deflections (x_ms). This seems odd, because large deflections due to a large loading are reached sooner than small deflections due to a small loading. This is a result of the non-linearity of the structure where it behaves stiffer for larger deflections. The reason that a single iso-deflection curve may still be used, is that the displacement does not appear linearly in $I_{s_{crit}}$ and $P_{m_{crit}}$. The values for $I_{s_{crit}}$ and $P_{m_{crit}}$ can be determined from the expression for the deformation energy (4) for the elastic membrane n = 3, and the energy principles (12) and (15). They appear proportional to $x_{m_{crit}}^2$ and x_{mcrit}^3, respectively.
Also combined deformation characteristics can be dealt with by the P-I technique in an approximate way. From the derived principles and the deformation energies, the expressions $I_{s_{crit}}$ and $P_{m_{crit}}$ can be established analytically. The hyperbola constant 'a' has to be determined by solving the equation of motion and is only valid for that specific $x_{m_{crit}}$, see (Erkel, 1988).

7 CONCLUSIONS

- General techniques on structural dynamics are ranked and considered.

- Analytical formulae for the maximum dynamic deflection of SDOF systems have been determined for the impulsive, dynamic and quasi-static loading realm.

- A general expression for the Dynamic Load Factor (DLF) has been derived for both linear and non-linear deformation characteristics. This factor relates the maximum dynamic deflection to the static deflection.

- The Pressure-Impulse (P-I) technique is considered with respect to applications beyond the scope of the DLF curve.

- For the transformation of beam structures to one-spring-mass systems, factors have been derived. These so-called transformation factors incorporate the influence of the deflection shape and the loading shape, which is a new feature.

- Convenient mathematical approximations have been found for the response curves.

- The derived formulae can be properly used in problems where an equal maximum dynamic deflection is required for different loading conditions and for parameter studies of structural response problems.

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To perform a quick analysis or a parameter study of structures under dynamic loads, the structure is often regarded as a Single Degree of Freedom (SDOF) system with one characteristic deflection.
At the TNO Prins Maurits Laboratory these systems are used as tools for the vulnerability analysis of the structures of weapon platforms under explosion loads. For naval ships, SDOF techniques are applied for the internal blast code DAMINEX, for the external blast algorithm CBD, for ship door research and for the research of underwater shock on the hull. Therefore research is done on SDOF systems, particularly for the response of stiffened panels under large deflections introducing non linear mechanisms.
This report gathers and explains some general techniques of structural dynamics for SDOF systems and serves as a basis for the application of the SDOF technique in vulnerability research. Analytical formulae for the maximum dynamic deflection have been determined, where the deflections are expressed in general terms for force, mass and stiffness. A general expression for the Dynamic Load Factor (DLF) is established for linear and non-linear deformation characteristics. Convenient mathematical approximations have been found for the response curves. New transformations factors have been derived. The curve for iso-damage based on the Pressure-Impulse technique is considered as a tool for the scope beyond the DLF. Parameter studies can be easily performed with the obtained results.

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Deformations	Models
Static Loading	Degrees of Freedom
Blast Loading	Dynamic Structural Analysis
Vulnerability	

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